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Estimations on numerically stable step-size for neutral delay differential systems with multiple delays

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Abstract

We derive two estimations of numerically stable step-size for systems of neutral delay differential equations with multiple delays. The stable step-size for numerical integration of NDDEs with multiple delays can be easily selected by means of the logarithmic norm and the spectral radius of certain matrices. Both explicit linear multistep methods and explicit Runge–Kutta methods are considered. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Consider the system of neutral delay differential equations (NDDEs) with multiple delays described by

$$\begin{aligned} \dot u(t) &= f(t, u(t), u(t-\tau_1), \ldots, u(t-\tau_m); \dot u(t-\tau_1), \ldots, \dot u(t-\tau_m)), \quad t \geqslant 0, \\ u(t) &= g(t), \quad -\tau_m \leqslant t \leqslant 0, \end{aligned} \tag{1}$$

where f and g are given vector-valued functions, τ_j is a given positive constant for $j = 1, \ldots, m$, $\tau_m > \tau_{m-1} > \cdots > \tau_1 > 0$, and $u(t)$ is the unknown vector-valued function.

We assume the existence of a unique solution of system (1). As in the case of ordinary differential equations (ODEs), the stability of numerical solution of NDDEs is crucial in obtaining good

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numerical approximations. As in the ODEs, the stability analysis is carried out through the linear system of NDDEs with multiple delays, i.e.,

$$\begin{aligned} \dot{u}(t) &= Lu(t)+\sum_{j=1}^{m}[M_j u(t-\tau_j)+N_j\dot{u}(t-\tau_j)], \quad t\geqslant 0,\\ u(t) &= g(t), \quad -\tau_m\leqslant t\leqslant 0. \end{aligned} \tag{2}$$

Here L, M_j and $N_j\in\mathbb{C}^{d\times d}$ are constant complex-valued matrices for $j=1,\dots,m$ and $\tau_m>\tau_{m-1}>\cdots>\tau_1>0$. Stability analyses of linear multistep methods and Runge–Kutta methods for system (2) in the case $\tau_j=j\tau$ have been given in [5, 7]. For earlier results of numerical solutions of Neutral equations and delay differential equations with many delays, see [1, 10, 11]; for recent results on numerical stability of Neutral and delay differential equations, see [4, 6].

The goal of the present paper is to extend the study of [5, 8, 9] and to give two practical ways to estimate the stable step-size for explicit linear multistep methods and explicit Runge–Kutta methods applied to system (2).

2. Numerical stability of (2)

In this section, we will review the results of [6]. We denote by $\lambda_j(A)$ the jth eigenvalue of $A\in\mathbb{C}^{n\times n}$ $(j=1,2,\dots,n)$. Consider the matrix

$$Q(v_1,\dots,v_m)=\left(I-\sum_{j=1}^{m}N_j v_j\right)^{-1}\left(L+\sum_{j=1}^{m}M_j v_j\right),$$

where $\sum_{j=1}^{m}\|N_j\|<1$, $v_j\in\mathbb{C}$ and $|v_j|\leqslant 1$. The following lemma states a sufficient condition for delay-independent stability of system (2).

Lemma 2.1 (Hu and Hu [6]). *System* (2) *is asymptotically stable if*

$$\sum_{j=1}^{m}\|N_j\|<1$$

and

$$\sup \operatorname{Re}\lambda_l(Q(v_1,\dots,v_m))<0$$

hold for $l=1,\dots,d$, *whenever* $v_j\in\mathbb{C}$ *and* $|v_j|\leqslant 1$ *for* $j=1,\dots,m$.

For the initial value problem of ODEs,

$$\dot{y}(t)=f(t,y(t)), \quad t\geqslant 0 \text{ and } y(0)=y_0,$$

a linear k-step method is given in a standard form as

$$\sum_{j=0}^{k}\alpha_j y_{n+j}=h\sum_{j=0}^{k}\beta_j f_{n+j}, \tag{3}$$

where h stands for the step-size and α_j, β_j are the formula parameters. Furthermore, a region $\mathbb{R}_{\mathrm{LM}}$ in the complex $\hat{h}$-plane is said to be the region of absolute stability if for all $\hat{h} \in \mathbb{R}_{\mathrm{LM}}$ the method is absolutely stable [12].

Consider method (3) applied to system (2). Let $t_l = lh$, $l \geqslant 0$, $h > 0$, and u_l be the numerical solution at the mesh points t_l. We have

$$\sum_{j=0}^{k} \alpha_j u_{n+j} = h \sum_{j=0}^{k} \beta_j v_{n+j} \tag{4}$$

and

$$v_{n+j} = L u_{n+j} + \sum_{v=1}^{m} [M_v u^h(t_{n+j} - \tau_v) + N_v v^h(t_{n+j} - \tau_v)] \tag{5}$$

for $n = 1, 2, \ldots, u^h(t) = g(t)$ and $t \leqslant 0$, and $u^h(t)$ with $t \geqslant 0$ is defined by

$$u^h(t_l + \delta h) = \sum_{j=-r}^{s} \hat{L}_j(\delta) u_{l+j},$$

for $0 \leqslant \delta < 1$, $l = 0, 1, \ldots,$ and

$$\hat{L}_j(\delta) = \prod_{k=-r, k \neq j}^{s} \frac{(\delta - k)}{(j - k)}. \tag{6}$$

Hence,

$$u^h(t_{n+j} - \tau_l) = \sum_{p=-r}^{s} \hat{L}_p(\delta_j) u_{n+j-l_l+p} \tag{7}$$

and

$$v^h(t_{n+j} - \tau_i) = \sum_{p=-r}^{s} \hat{L}_p(\delta_j) v_{n+j-l_i+p}, \tag{8}$$

where $r, s \geqslant 0$ are integers and $r \leqslant s \leqslant r+2$, $l_j = [\tau_j h^{-1}]$, $\delta_j = l_j - \tau_j h^{-1}$, $0 \leqslant \delta_j < 1$ for $j = 1, \ldots, m$, $l_m \geqslant \cdots \geqslant l_1 \geqslant s+1$; here $[q]$ denotes the smallest integer that is greater than or equal to $q \in \mathbb{R}$.

A characterization of the region of absolute stability in NDDEs with multiple delays is given by [6].

Lemma 2.2. *If*

(i) *the assumptions of Lemma* 2.1 *hold, and*

(ii) $h\lambda_l(Q(v_1, \ldots, v_m)) \in \mathbb{R}_{\mathrm{LM}}$ *for* $l = 1, \ldots, d$ *and* $v_j \in \mathbb{C}$ *such as* $|v_j| \leqslant 1$ *for* $j = 1, \ldots, m$,

(iii) $r \leqslant s \leqslant r+2$,

then the linear multistep method in (4)–(8) *applied to* (2) *is asymptotically stable.*

Next, we consider an application of $\hat{s}$-stage Runge–Kutta (RK in short) method in the ODE case to system (2). Denote the stage values of the RK formula by $k_{n,i}$. Let $t_l = lh$, $l \geqslant 0$, $h > 0$, and u_l be the numerical solution at the mesh points t_l. We obtain the natural RK scheme for system (2) as follows:

$$k_{n,i} = hL\left(u_n + \sum_{j=1}^{\hat{s}} a_{ij}k_{n,j}\right) + h\sum_{\hat{\imath}=1}^{m} M_{\hat{\imath}}\left(u_{n-l_{\hat{\imath}}+\delta_{\hat{\imath}}} + \sum_{j=1}^{\hat{s}} a_{ij}k_{n-l_{\hat{\imath}}+\delta_{\hat{\imath}},j}\right) + \sum_{\hat{\imath}=1}^{m} N_{\hat{\imath}}k_{n-l_{\hat{\imath}}+\delta_{\hat{\imath}},i}, \tag{9}$$

$$u_{n+1} = u_n + \sum_{i=1}^{\hat{s}} b_i k_{n,i}. \tag{10}$$

Here $i = 1, 2, \ldots, \hat{s}$, a_{ij} and b_i denote the parameters of the underlying Runge–Kutta method. For $n = 1, 2, \ldots, u_{n-l_i+\delta_i} = g((n-l_i+\delta_i)h)$ for $n-l_i+\delta_i \leqslant 0, u_{n-l_i+\delta_i}$ and $k_{n-l_i+\delta_i,j}$ with $n-l_i+\delta_i \geqslant 0$ are defined by the following respective interpolations:

$$u_{n-l_i+\delta_i} = \sum_{p=-r}^{s} \hat{L}_p(\delta_i)u_{n-l_i+p}, \tag{11}$$

$$k_{n-l_i+\delta_i,j} = \sum_{p=-r}^{s} \hat{L}_p(\delta_i)k_{n-l_i+p,j} \tag{12}$$

and

$$L_p(\delta) = \prod_{k=-r,k\neq p}^{s} \frac{(\delta-k)}{(p-k)} \tag{13}$$

for $0 \leqslant \delta < 1$, $i = 1, \ldots, m$, $j = 1, \ldots, \hat{s}$, where $r, s \geqslant 0$ are integers, $r \leqslant s \leqslant r+2$ and $l_i = [\tau_i h^{-1}]$, $\delta_i = l_i - \tau_i h^{-1}$, $0 \leqslant \delta_i < 1$ for $i = 1, \ldots, m$, $l_m \geqslant \cdots \geqslant l_i \geqslant s+1$. Here $[q]$ denotes the smallest integer that is greater than or equal to $q \in \mathbb{R}$.

Let $\mathbb{R}_{\mathrm{RK}}$ denote the region of absolute stability of the RK method in the ODE case [12]. The following are conditions for numerical stability of an explicit natural RK for system (2).

Lemma 2.3 (Hu and Hu [6]). *Assume that*
(i) *the assumptions of Lemma* 2.1 *hold,*
(ii) $h\lambda_i(Q(v_1, \ldots, v_m)) \in \mathbb{R}_{\mathrm{RK}}$ *for all* $i = 1, \ldots, d$ *and* $v_i \in \mathbb{C}$ *such as* $|v_i| \leqslant 1$ *for* $i = 1, \ldots, m$,
(iii) $r \leqslant s \leqslant r+2$.
Then the natural RK *scheme in* (9)–(13) *for system* (2) *is asymptotically stable.*

In view of Lemmas 2.2 and 2.3, the eigenvalues of $Q(v_1, \ldots, v_m)$ with $|v_i| \leqslant 1$ govern the stability of LM and RK methods. But it is difficult to select a stable step-size h by means of Lemmas 2.2 and 2.3 which require computation of $\lambda_i(Q(v_1, \ldots, v_m))$, for $i = 1, 2, \ldots, d$ and $|v_j| \leqslant 1$ $(j = 1, \ldots, m)$. In the following sections, on the basis of Lemmas 2.2 and 2.3, two simple estimations for the stability regions for explicit LM and RK methods are derived by means of the logarithmic norm and the spectral radius.

3. Estimation on numerically stable step-size for (2) via logarithmic norm

Lemma 3.1 (Lancaster and Tismenetsky [13]). *Let $W \in \mathbb{C}^{n\times n}$. If $\rho(W)<1$, where $\rho(W)$ is the spectral radius of the matrix W. Then $(I+W)^{-1}$ exists and*

$$(I+W)^{-1}=I-W+W^2-\cdots=I+(I+W)^{-1}(-W).$$

Also, if $\|W\|<1$ then

$$\|(I+W)^{-1}\| \leqslant \frac{1}{1-\|W\|}.$$

Let $\mu(W)$ denote the logarithmic matrix norm, that is,

$$\mu(W)=\lim_{\Delta\to 0^+}\frac{\|I+\Delta W\|-1}{\Delta}.$$

Lemma 3.2 (Desoer and Vidyasagar [2]). *For each eigenvalue $\lambda_j(W)$ of $W\in\mathbb{C}^{d\times d}$, the inequality $-\mu(-W)\leqslant \operatorname{Re}\lambda_j(W)\leqslant \mu(W)$ holds.*

Definition 3.3. The real scalar quantities are defined as

$$X=\frac{\sum_{j=1}^{m}\|N_jL\|+\sum_{j=1}^{m}(\sum_{k=1}^{m}\|N_jM_k\|)}{1-\sum_{j=1}^{m}\|N_j\|},$$

$$E_1=-\mu(-L)-\sum_{j=1}^{m}\|M_j\|-X,$$

$$E_2=\min\{0,l\},$$

where $l=\mu(L)+\sum_{j=1}^{m}\|M_j\|+X$,

$$F_1=-\mu(\mathrm{i}L)-\sum_{j=1}^{m}\|M_j\|-X$$

and

$$F_2=\mu(-\mathrm{i}L)+\sum_{j=1}^{m}\|M_j\|+X,$$

where $\mathrm{i}^2=-1$.

Making use of these, we obtain the following estimations.

Theorem 3.4. *Assume that the conditions of Lemma* 2.1 *hold. Then the eigenvalues of the matrix* $Q(v_1,\dots,v_m)(v_j\in\mathbb{C}$ *and* $|v_j|\leqslant 1)$ *satisfy the following estimations:*

$$E_1\leqslant \operatorname{Re}\lambda_j(Q(v_1,\dots,v_m))\leqslant E_2$$

and

$$F_1\leqslant \operatorname{Im}\lambda_j(Q(v_1,\dots,v_m))\leqslant F_2.$$

Proof. From Section 2 we have

$$Q(v_1,\dots,v_m)=\left(I-\sum_{j=1}^{m}N_jv_j\right)^{-1}\left(L+\sum_{j=1}^{m}M_jv_j\right)$$

for $\sum_{j=1}^{m}\|N\|<1$, $v_j\in\mathbb{C}$ and $|v_j|\leqslant 1$.

Let $\hat{N}=\sum_{j=1}^{m}N_jv_j$, and $\hat{M}=\sum_{j=1}^{m}M_jv_j$.

According to Lemma 3.2, we have the inequality

$$\operatorname{Re}\lambda_j(Q(v_1,\dots,v_m))\leqslant\mu(Q(v_1,\dots,v_m))$$

for $\sum_{j=1}^{m}|N_j|<1$, $v_j\in\mathbb{C}$ and $|v_j|\leqslant 1$. In the following, we show that

$$\mu(Q(v_1,v_2,\dots,v_m))\leqslant\mu(L)+\sum_{j=1}^{m}\|M_j\|+X$$

holds. From Lemmas 3.1 and 3.2, for $v_j\in\mathbb{C}$ and $|v_j|\leqslant 1$, we obtain

$$\begin{aligned}\mu(Q(v_1,v_2,\dots,v_m)) &= \mu[(I+\hat{N}+\hat{N}^2+\cdots)(L+\hat{M})]\\ &= \mu[(L+\hat{M})+(\hat{N}+\hat{N}^2+\cdots)(L+\hat{M})]\\ &\leqslant \mu(L)+\mu(\hat{M})+\mu[(\hat{N}+\hat{N}^2+\cdots)(L+\hat{M})]\\ &\leqslant \mu(L)+\|\hat{M}\|+\|(I+\hat{N}+\hat{N}^2+\cdots)(\hat{N}L+\hat{N}\hat{M})\|\\ &\leqslant \mu(L)+\sum_{j=1}^{m}\|M_j\|+(\|I\|+\|\hat{N}\|+\|\hat{N}2\|+\cdots)(\|\hat{N}L\|+\|\hat{N}\hat{M}\|)\\ &\leqslant \mu(L)+\sum_{j=1}^{m}\|M_j\|+\left(\|I\|+\sum_{j=1}^{m}\|N_j\|+\left(\sum_{j=1}^{m}\|N_j\|\right)^2+\cdots\right)\end{aligned}$$

$$\times\left(\sum_{j=1}^{m}\|N_jL\|+\sum_{j=1}^{m}\left(\sum_{k=1}^{m}\|N_jM_k\|\right)\right)$$

$$=\mu(L)+\sum_{j=}^{m}\|M_j\|+\frac{(\sum_{j=1}^{m}\|N_jL\|+\sum_{j=1}^{m}(\sum_{k=1}^{m}\|N_jM_k\|))}{1-\sum_{j=1}^{m}\|N_j\|}$$

$$=\mu(L)+\sum_{j=1}^{m}\|M_j\|+X=l.$$

Since the conditions of Lemma 2.1 hold, we have

$$\operatorname{Re}\lambda_j(Q(v_1,v_2,\ldots,v_m))\leqslant E_2=\min\{0,l\}.$$

Thus Lemma 3.2 yields

$$-\mu(-Q(v_1,v_2,\ldots,v_m))\leqslant \operatorname{Re}\lambda_j(Q(v_1,v_2,\ldots,v_m)),$$

which in turn yields

$$E_1\leqslant \operatorname{Re}\lambda_j(Q(v_1,v_2,\ldots,v_m)).$$

Since

$$\operatorname{Im}\lambda_j(Q(v_1,v_2,\ldots,v_m))=\operatorname{Re}\lambda_j(-\mathrm{i}Q(v_1,v_2,\ldots,v_m)),$$

we obtain

$$-\mu(\mathrm{i}Q(v_1,v_2,\ldots,v_m))\leqslant \operatorname{Re}\lambda_j(-\mathrm{i}Q(v_1,v_2,\ldots,v_m))\leqslant\mu(-\mathrm{i}Q(v_1,v_2,\ldots,v_m)).$$

To demonstrate the inequality

$$F_1\leqslant \operatorname{Im}\lambda_j(Q(v_1,v_2,\ldots,v_m))\leqslant F_2,$$

we repeat similar calculations as for $\operatorname{Re}\lambda_j(Q(v_1,v_2,\ldots,v_m))$. Thus the proof is completed.

Definition 3.5. Assume that $\sum_{j=1}^{m}\|N_j\|<1$. We define

$$D(h)=h\mathbb{G},$$

where h is the step-size and

$$\mathbb{G}=\{z=x+\mathrm{i}y;\ E_1\leqslant x\leqslant E_2,\ F_1\leqslant y\leqslant F_2\}.$$

Now we state the main result which gives a simple way to find a numerically stable step-size for system (2).

Theorem 3.6. *Assume that the conditions of Lemma* 2.1 *and* $r \leqslant s \leqslant r+2$ *hold.*
(i) *If* $D(h) \subset \mathbb{R}_{\mathrm{LM}}$ *for some positive* h, *then the explicit linear multistep method* (4)–(8) *for system* (2) *is asymptotically stable.*
(ii) *If* $D(h) \subset \mathbb{R}_{\mathrm{RK}}$ *for some positive* h, *then the explicit natural Runge–Kutta method* (9)–(13) *is asymptotically stable.*

Proof. In the case of the LM method, due to Theorem 3.4, we have

$$h\lambda_j(Q(v_1, v_2, \dots, v_m)) \subset D(h) \subset \mathbb{R}_{\mathrm{LM}},$$

which implies the stability by virtue of Lemma 2.2. The proof for the RK is similar. □

4. Estimation on numerically stable step-size for (2) via spectral radius

In this section we need the following definitions and lemmas. Let $W \in \mathbb{C}^{n\times n}$ with elements w_{jk} and $|W|$ denote the nonnegative matrix in $R^{n\times n}$ with elements $|w_jk|$. Let $W = \{w_{jk}\}$ and $V = \{v_{jk}\} \in R^{n\times n}$. We say $|W| \leqslant V$ if and only if $|w_jk| \leqslant v_{jk}$ for all pairs of (j,k).

Lemma 4.1 (Lancaster and Tismenetsky [13]). *Let* $W \in \mathbb{C}^{n\times n}$ *and* $V \in \mathbb{R}^{n\times n}$. *If* $|W| \leqslant V$, *then* $\rho(W) \leqslant \rho(V)$, *where* $\rho(W)$ *and* $\rho(V)$ *denote the spectral radii of* W *and* V, *respectively.*

Definition 4.2. We define

$$Y = \left(I - \sum_{j=1}^{m} |N_j|\right)^{-1} \left(\sum_{j=1}^{m} |N_j L| + \sum_{j=1}^{m} \left(\sum_{k=1}^{m} |N_j M_k|\right)\right)$$

and

$$\hat{Y} = |L| + \sum_{j=1}^{m} |M_j| + Y.$$

Theorem 4.3. *If* $\sum_{j=1}^{m} \|N_j\| < 1$, *then*

$$\rho(Q(v_1, v_2, \dots, v_m)) \leqslant \rho(\hat{Y})$$

for $v_j \in \mathbb{C}$ *and* $|v_j| \leqslant 1, \quad j = 1, \dots, m.$

Proof. According to Lemma 4.1 and

$$\rho\left(\sum_{j=1}^{m} |N_j|\right) \leqslant \sum_{j=1}^{m} \|N_j\| < 1,$$

we have

$$\det\left(I - \sum_{j=1}^{m} |N_j|\right) \neq 0.$$

Set $\hat{M} = \sum M_j v_j$ and $\hat{N} = \sum N_j v_j$ where $|v_j| \leqslant 1$, $j = 1, \ldots, m$. Thus $|\hat{M}| \leqslant \sum_{j=1}^{m} |M_j|$ and $|\hat{N}| \leqslant \sum_{j=1}^{m} |N_j|$. From Lemma 4.1, we obtain

$$\begin{aligned}
|Q(v_1, v_2, \ldots, v_m)| &= |(I - \hat{N})^{-1}(L + \hat{M})| \\
&= |(I + \hat{N} + \hat{N}^2 + \cdots)(L + \hat{M})| \\
&\leqslant |L| + |\hat{M}| + (|\hat{N} + \hat{N}^2 + \cdots)(L + \hat{M})| \\
&= |L| + |\hat{M}| + |(I + \hat{N} + \hat{N}^2 + \cdots)(\hat{N}L + \hat{N}\hat{M})| \\
&\leqslant |L| + \sum_{j=1}^{m} |M_j| + (|I| + |\hat{N}| + |\hat{N}^2| + \cdots)(|\hat{N}L| + |\hat{N}\hat{M}|) \\
&\leqslant |L| + \sum_{j=1}^{m} |M_j| + \left(|I| + \sum_{j=1}^{m} |N_j| + \left(\sum_{j=1}^{m} |N_j|^2 + \cdots\right)\right) \\
&\quad \times \left(\sum_{j=1}^{m} |N_j L| + \left(\sum_{j=1}^{m} \sum_{k=1}^{m} |N_j M_k|\right)\right) \\
&= |L| + \sum_{j=1}^{m} |M_j| + \left(I - \sum_{j=1}^{m} |N_j|\right)^{-1} \left(\sum_{j=1}^{m} |N_j L| + \sum_{j=1}^{m} \left(\sum_{k=1}^{m} |N_j M_k|\right)\right) \\
&= |L| + \sum_{j=1}^{m} |M_j| + Y = \hat{Y}.
\end{aligned} \tag{15}$$

According to Lemma 4.1, the proof is complete. □

We need the following definition for the next result

Definition 4.4. We define the region $K(h)$ in the complex plane as

$$K(h) = \left\{ (\gamma', \theta) \colon 0 \leqslant \gamma' \leqslant h\rho(\hat{Y}), \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2} \right\},$$

where h is the step-size and $\rho(\hat{Y})$ is the spectral radius of the matrix $\hat{Y}$ which is defined in Definition 4.2.

Theorem 4.5. *Assume that the conditions of Lemma* 2.1 *and* $r \leqslant s \leqslant r+2$ *hold.*

(i) *If* $K(h) \subset \mathbb{R}_{LM}$ *for some positive* h, *then the explicit linear multistep method* (4)–(8) *for system* (2) *is asymptotically stable for these choices of* h.

(ii) *If* $K(h) \subset \mathbb{R}_{RK}$ *for some positive* h, *then the explicit natural Runge–Kutta method* (9)–(13) *is asymptotically stable for these choices of* h.

Proof. The proof is similar to the proof of Theorem 3.6.

Remark 4.6. Theorems 4.5 and Theorem 3.6 have shown a practical way to find a stable step-size h. Obviously, the choice of the step size h by $h\lambda_j(Q(v_1, v_2, \ldots, v_m))$ is sharper than the step size h selected $K(h)$ or $D(h)$.

5. Examples

In this section we present several examples using the main results of this paper.

Consider system (2) with $m=2$, i.e.,

$$\dot{u}(t) = Lu(t) + \sum_{j=1}^{2} [M_j u(t-\tau_j) + N_j \dot{u}(t-\tau_j)], \quad t \geqslant 0 \tag{16}$$

and

$$\tau_2 > \tau_1 > 0.$$

Example 1. Let

$$L = \begin{bmatrix} -95 & 0 & 2 \\ -1 & -95 & 0 \\ 1 & 1 & -95 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 2 \\ 3 & 0 & 5 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} \frac{1}{50} & \frac{1}{25} & \frac{1}{50} \\ \frac{-1}{25} & \frac{3}{50} & \frac{1}{50} \\ \frac{1}{25} & \frac{1}{10} & \frac{7}{50} \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 3 \\ 4 & 6 & 1 \end{bmatrix},$$

and

$$N_2 = \begin{bmatrix} \frac{1}{25} & \frac{1}{10} & \frac{3}{50} \\ \frac{-1}{50} & 0 & \frac{3}{50} \\ \frac{3}{50} & \frac{1}{25} & \frac{2}{25} \end{bmatrix}.$$

In this example $\|N_1\|_1 = \frac{1}{5}$, $\|N_2\|_1 = \frac{1}{5}$.

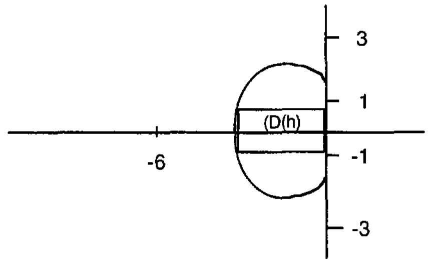

Fig. 1.

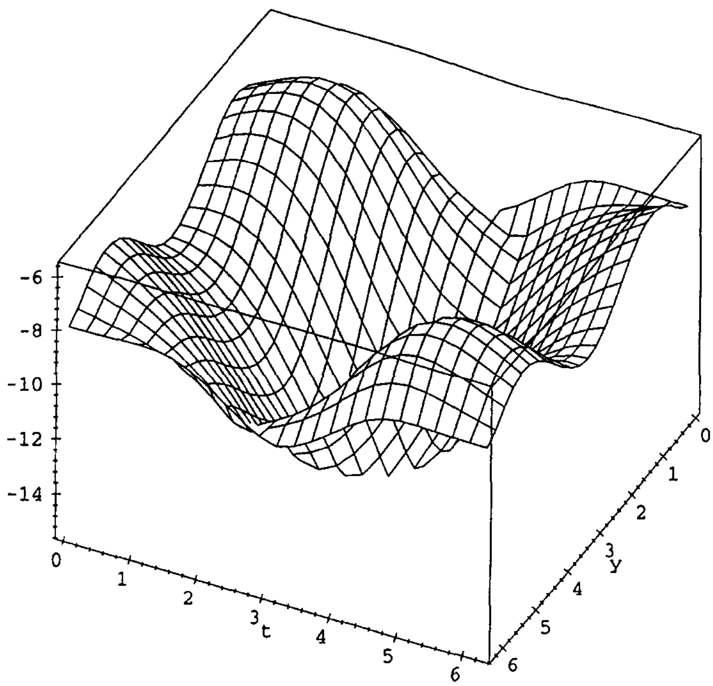

Fig. 2.

By direct calculation we obtain

$$E_2 = -4, \quad E_1 = -186, \quad F_1 = 91, \quad F_2 = -91.$$

Thus

$$\mathbb{G} = \{z = x + \mathrm{i}y\colon\ -186 \leqslant x \leqslant -4,\ -91 \leqslant y \leqslant 91\}$$

and

$$D(h) = h\mathbb{G},$$

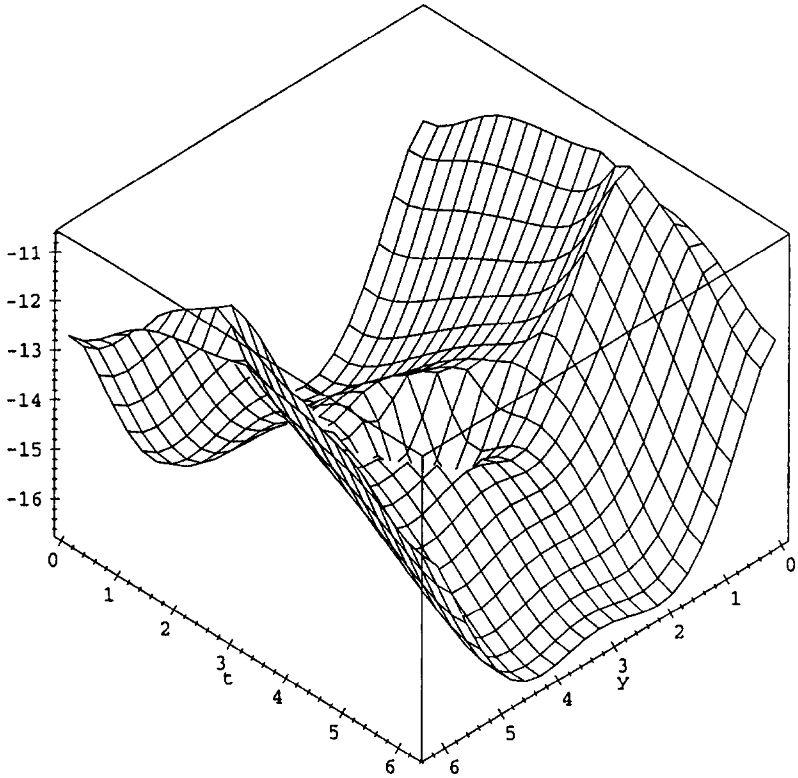

Fig. 3.

and the stepsize h is determined by $h < \tau_1$ and $h\mathbb{G} \subset \mathbb{R}_{\mathrm{LM}}$ or $h\mathbb{G} \subset \mathbb{R}_{\mathrm{RK}}$. We use the Adams–Moulton method of order 4 (see [3] and Fig. 1). For system (2) we obtain the numerically stable step-size to be $h < 0.01$ and $0 < h < \tau_1$.

In this example

$$\rho(\hat{Y}) = 136.0379$$

and

$$K(h) = \{(\gamma', \theta)\colon\ 0 \leqslant \gamma' \leqslant 136.0379h,\ \tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{3}{2}\pi\},$$

and a numerically stable step-size h is determined by $h < \tau_1$ and $K(h) \subset \mathbb{R}_{\mathrm{LM}}$ for a linear k-step method applied to system (2) and $K(h) \subset \mathbb{R}_{\mathrm{RK}}$ for RK method applied to system (2). In the case of the Adams–Bashforth method of order 3 applied to system (2), a numerically stable step-size is determined by $0 < h < \tau_1$ and $h < 0.01$; see [3].

Notice that if we use

$$\mu(L) + \sum_{j=1}^{2} \|M_j\|_1 \frac{\sum_{j=1}^{2} \|N_j\|_1 \|L\|_1 + \sum_{j=1}^{2} \sum_{k=1}^{2} \|N_j\|_1 \|M_k\|_1}{1 - \sum_{j=1}^{2} \|N_j\|_1},$$

as in [3], we get

$$\mu(Q(v_1, v_2)) \leqslant 1,$$

which is inconclusive, and asymptotic stability is not guaranteed.

Example 2. Consider system (2) again with the following matrices:

$$L = \begin{bmatrix} -12 & 1 \\ 2 & -12 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \qquad N_1 = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ -\frac{1}{5} & \frac{3}{10} \end{bmatrix},$$

$$N_2 = \begin{bmatrix} \frac{1}{5} & \frac{1}{2} \\ -\frac{1}{10} & 0 \end{bmatrix}.$$

In this example $\|N_1\|_1 + \|N_2\|_1 = 1$ and Theorems 3 and 5 cannot be applied. However, considering the real parts of the eigenvalues of $Q(v_1, v_2)$, we can see in Figs. 2 and 3 that

$$\operatorname{Re}\lambda_1(v_1, v_2) < 0, \qquad \operatorname{Re}\lambda_2(v_1, v_2) < 0,$$

respectively. The surfaces are derived in this example from Maple. Therefore, system (16) with the above matrices is asymptotically stable, and for $h > 0$ sufficiently small, the linear multistep method (4)–(8) and RK method (9)–(13) applied to (16) are asymptotically stable.

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